Probabilities and Quantiles

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2021-09-05

Introduction

This vignette details how the functions `dml()`, `pml()`, `qml()` and `rml()` are evaluated using the Mittag-Leffler function `mlf()` and functions from the package `stabledist`. Evaluation of the Mittag-Leffler function relies on the algorithm by Garrappa (2015).

Mittag-Leffler function Write $E_{\alpha,\beta}(z)$ for the two-parameter Mittag-Leffler function, and $E_\alpha(z) := E_{\alpha,1}(z)$ for the one-parameter Mittag-Leffler function. One has

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta + \alpha k)}, \quad \alpha \in \mathbb{C}, \Re(\alpha) > 0, z \in \mathbb{C},$$

see Haubold, Mathai, and Saxena (2011).

First type Mittag-Leffler distribution

`pml()` The cumulative distribution function at unit scale is (see Haubold, Mathai, and Saxena (2011))

$$F(y) = 1 - E_\alpha(-y^\alpha)$$

`dml()` The probability density function at unit scale is (see Haubold, Mathai, and Saxena (2011))

$$f(y) = \frac{d}{dy} F(y) = y^{\alpha-1} E_{\alpha,\alpha}(-y^\alpha)$$

`qml()` The quantile function `qml()` is calculated by numeric inversion of the cumulative distribution function `pml()` using `stats::uniroot()`.

`rml()` Mittag-Leffler random variables Z are generated as the product of a stable random variable Y with Laplace Transform $\exp(-s^\alpha)$ (using the package `stabledist`) and $X^{1/\alpha}$ where X is a unit exponentially distributed random variable, see Haubold, Mathai, and Saxena (2011).

Second type Mittag-Leffler distribution

Meerschaert and Scheffler (2004) introduce the inverse stable subordinator, a stochastic process $E(t)$. The random variable $E := E(1)$ has unit scale Mittag-Leffler distribution of second type, see the equation under Remark 3.1. By Corollary 3.1, E is equal in distribution to $Y^{-\alpha}$:

$$E \stackrel{d}{=} Y^{-\alpha},$$

where Y is a sum-stable randomvariable as above.

`pml()` Using `stabledist`, we can hence calculate the cumulative distribution function of E:

$$\mathbf{P}[E \leq q] = \mathbf{P}[Y^{-\alpha} \leq q] = \mathbf{P}[Y \geq q^{-1/\alpha}]$$

`dml()` The probability density function is evaluated using the formula

$$f(x) = \frac{1}{\alpha} x^{-1-1/\alpha} f_Y(x^{-1/\alpha})$$

where $f_Y(x)$ is the probability density of the stable random variable Y.

`qml()` Let $q = (F_Y^{-1}(1-p))^{-\alpha}$, where $p \in (0,1)$ and F_Y^{-1} denotes the quantile function of Y, implemented in `stabledist`. Then one confirms

$$F_Y(q^{-1/\alpha}) = 1 - p \Rightarrow \mathbf{P}[Y \geq q^{-1/\alpha}] = p \Rightarrow \mathbf{P}[Y^{-\alpha} \leq q] = p$$

which means $F_E(q) = p$.

`rml()` Mittag-Leffler random variables E of second type are directly simulated as $Y^{-\alpha}$, using `stabledist`.

References

Garrappa, Roberto. 2015. "Numerical Evaluation of Two and Three Parameter Mittag-Leffler Functions." *SIAM J. Numer. Anal.* 53 (3): 1350–69. https://doi.org/10.1137/140971191.

Haubold, H. J., A. M. Mathai, and R. K. Saxena. 2011. "Mittag-Leffler Functions and Their Applications." *J. Appl. Math.* 2011: 1–51. https://doi.org/10.1155/2011/298628.

Meerschaert, Mark M, and Hans-Peter Scheffler. 2004. "Limit Theorems for Continuous-Time Random Walks with Infinite Mean Waiting Times." *J. Appl. Probab.* 41 (3): 623–38. https://doi.org/10.1239/jap/1091543414.