The SPDE model with transparent barriers

Elias T Krainski

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The transparent barrier model

This model considers an SPDE over a domain Ω which is partitioned into k subdomains Ω_d, $d \in \{1, \ldots, k\}$, where $\cup_{d=1}^k \Omega_d = \Omega$. A common marginal variance is assumed but the range can be particular to each Ω_d, r_d.

From Bakka et al. (2019), the precision matrix is

$$\mathbf{Q} = \frac{1}{\sigma^2}\mathbf{R}\tilde{\mathbf{C}}^{-1}\mathbf{R} \text{ for } \mathbf{R}_r = \mathbf{C} + \frac{1}{8}\sum_{d=1}^{k} r_d^2 \mathbf{G}_d, \quad \tilde{\mathbf{C}}_r = \frac{\pi}{2}\sum_{d=1}^{k} r_d^2 \tilde{\mathbf{C}}_d$$

where σ^2 is the marginal variance. The Finite Element Method - FEM matrices: $\mathbf{C}$, defined as

$$\mathbf{C}_{i,j} = \langle \psi_i, \psi_j \rangle = \int_\Omega \psi_i(\mathbf{s})\psi_j(\mathbf{s})\partial\mathbf{s},$$

computed over the whole domain, while $\mathbf{G}_d$ and $\tilde{\mathbf{C}}_d$ are defined as a pair of matrices for each subdomain

$$(\mathbf{G}_d)_{i,j} = \langle 1_{\Omega_d}\nabla\psi_i, \nabla\psi_j \rangle = \int_{\Omega_d} \nabla\psi_i(\mathbf{s})\nabla\psi_j(\mathbf{s})\partial\mathbf{s} \text{ and } (\tilde{\mathbf{C}}_d)_{i,i} = \langle 1_{\Omega_d}\psi_i, 1 \rangle = \int_{\Omega_d} \psi_i(\mathbf{s})\partial\mathbf{s}.$$

In the case when $r = r_1 = r_2 = \ldots = r_k$ we have $\mathbf{R}_r = \mathbf{C} + \frac{r^2}{8}\mathbf{G}$ and $\tilde{\mathbf{C}}_r = \frac{\pi r^2}{2}\tilde{\mathbf{C}}$ giving

$$\mathbf{Q} = \frac{2}{\pi\sigma^2}\left(\frac{1}{r^2}\mathbf{C}\tilde{\mathbf{C}}^{-1}\mathbf{C} + \frac{1}{8}\mathbf{C}\tilde{\mathbf{C}}^{-1}\mathbf{G} + \frac{1}{8}\mathbf{G}\tilde{\mathbf{C}}^{-1}\mathbf{C} + \frac{r^2}{64}\mathbf{G}\tilde{\mathbf{C}}^{-1}\mathbf{G}\right)$$

which coincides with the stationary case in Lindgren and Rue (2015), when using $\tilde{\mathbf{C}}$ in place of $\mathbf{C}$.

Implementation

In practice we define r_d as $r_d = p_d r$, for known $p_1, \ldots, p_k$ constants. This gives

$$\tilde{\mathbf{C}}_r = \frac{\pi r^2}{2}\sum_{d=1}^{k} p_d^2 \tilde{\mathbf{C}}_d = \frac{\pi r^2}{2}\tilde{\mathbf{C}}_{p_1,\ldots,p_k} \text{ and } \frac{1}{8}\sum_{d=1}^{k} r_d^2 \mathbf{G}_d = \frac{r^2}{8}\sum_{d=1}^{k} p_d^2 \tilde{\mathbf{G}}_d = \frac{r^2}{8}\tilde{\mathbf{G}}_{p_1,\ldots,p_k}$$

where $\tilde{\mathbf{C}}_{p_1,\ldots,p_k}$ and $\tilde{\mathbf{G}}_{p_1,\ldots,p_k}$ are pre-computed.

References

Bakka, H., J. Vanhatalo, J. Illian, D. Simpson, and H. Rue. 2019. "Non-Stationary Gaussian Models with Physical Barriers." *Spatial Statistics* 29 (March): 268–88. https://doi.org/https://doi.org/10.1016/j.spasta.2019.01.002.

Lindgren, Finn, and Havard Rue. 2015. "Bayesian Spatial Modelling with R-INLA." *Journal of Statistical Software* 63 (19): 1–25.