

**A Handbook of Statistical Analyses Using R**  
— **3rd Edition**

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## Density Estimation: Erupting Geysers and Star Clusters

### 8.1 Introduction

### 8.2 Density Estimation

The three kernel functions are implemented in R as shown in lines 1–3 of Figure 8.1. For some grid  $\mathbf{x}$ , the kernel functions are plotted using the R statements in lines 5–11 (Figure 8.1).

The kernel estimator  $\hat{f}$  is a sum of ‘bumps’ placed at the observations. The kernel function determines the shape of the bumps while the window width  $h$  determines their width. Figure 8.2 (redrawn from a similar plot in Silverman, 1986) shows the individual bumps  $n^{-1}h^{-1}K((x-x_i)/h)$ , as well as the estimate  $\hat{f}$  obtained by adding them up for an artificial set of data points

```
R> x <- c(0, 1, 1.1, 1.5, 1.9, 2.8, 2.9, 3.5)
```

```
R> n <- length(x)
```

For a grid

```
R> xgrid <- seq(from = min(x) - 1, to = max(x) + 1, by = 0.01)
```

on the real line, we can compute the contribution of each measurement in  $\mathbf{x}$ , with  $h = 0.4$ , by the Gaussian kernel (defined in Figure 8.1, line 3) as follows;

```
R> h <- 0.4
```

```
R> bumps <- sapply(x, function(a) gauss((xgrid - a)/h)/(n * h))
```

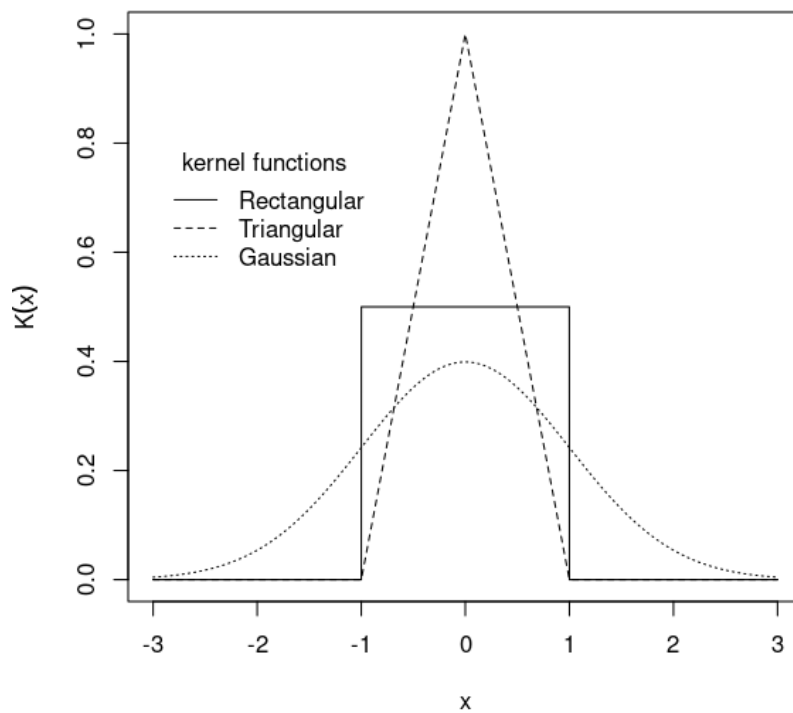
A plot of the individual bumps and their sum, the kernel density estimate  $\hat{f}$ , is shown in Figure 8.2.

### 8.3 Analysis Using R

#### 8.3.1 A Parametric Density Estimate for the Old Faithful Data

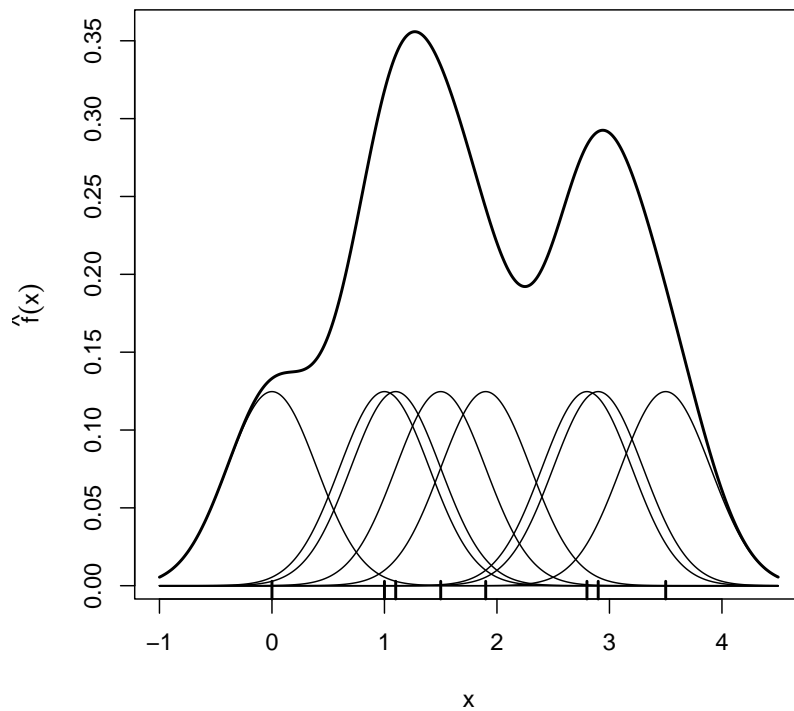
```
R> logL <- function(param, x) {
+   d1 <- dnorm(x, mean = param[2], sd = param[3])
+   d2 <- dnorm(x, mean = param[4], sd = param[5])
+   -sum(log(param[1] * d1 + (1 - param[1]) * d2))
+ }
R> startparam <- c(p = 0.5, mu1 = 50, sd1 = 3, mu2 = 80, sd2 = 3)
R> opp <- optim(startparam, logL, x = faithful$waiting,
```

```
1 R> rec <- function(x) (abs(x) < 1) * 0.5
2 R> tri <- function(x) (abs(x) < 1) * (1 - abs(x))
3 R> gauss <- function(x) 1/sqrt(2*pi) * exp(-(x^2)/2)
4 R> x <- seq(from = -3, to = 3, by = 0.001)
5 R> plot(x, rec(x), type = "l", ylim = c(0,1), lty = 1,
6 +       ylab = expression(K(x)))
7 R> lines(x, tri(x), lty = 2)
8 R> lines(x, gauss(x), lty = 3)
9 R> legend(-3, 0.8, legend = c("Rectangular", "Triangular",
10 + "Gaussian"), lty = 1:3, title = "kernel functions",
11 +       bty = "n")
```



**Figure 8.1** Three commonly used kernel functions.

```
1 R> plot(xgrid, rowSums(bumps), ylab = expression(hat(f)(x)),
2 +       type = "l", xlab = "x", lwd = 2)
3 R> rug(x, lwd = 2)
4 R> out <- apply(bumps, 2, function(b) lines(xgrid, b))
```



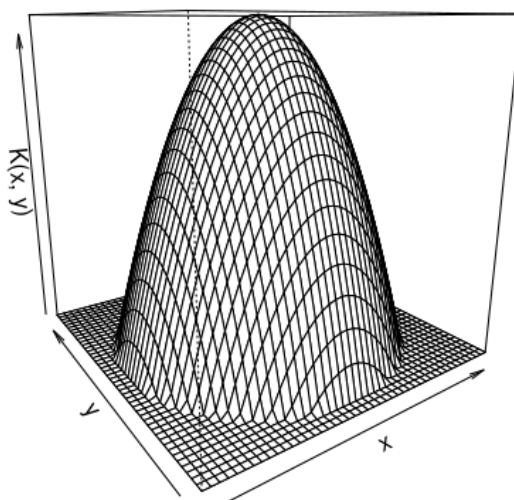
**Figure 8.2** Kernel estimate showing the contributions of Gaussian kernels evaluated for the individual observations with bandwidth  $h = 0.4$ .

```
+         method = "L-BFGS-B",
+         lower = c(0.01, rep(1, 4)),
+         upper = c(0.99, rep(200, 4)))
```

```

R> epa <- function(x, y)
+   ((x^2 + y^2) < 1) * 2/pi * (1 - x^2 - y^2)
R> x <- seq(from = -1.1, to = 1.1, by = 0.05)
R> epavals <- sapply(x, function(a) epa(a, x))
R> persp(x = x, y = x, z = epavals, xlab = "x", ylab = "y",
+       zlab = expression(K(x, y)), theta = -35, axes = TRUE,
+       box = TRUE)

```



**Figure 8.3** Epanechnikov kernel for a grid between  $(-1.1, -1.1)$  and  $(1.1, 1.1)$ .

```

R> opp
$par
  p    mu1    sd1    mu2    sd2
0.361 54.612  5.872 80.093  5.867

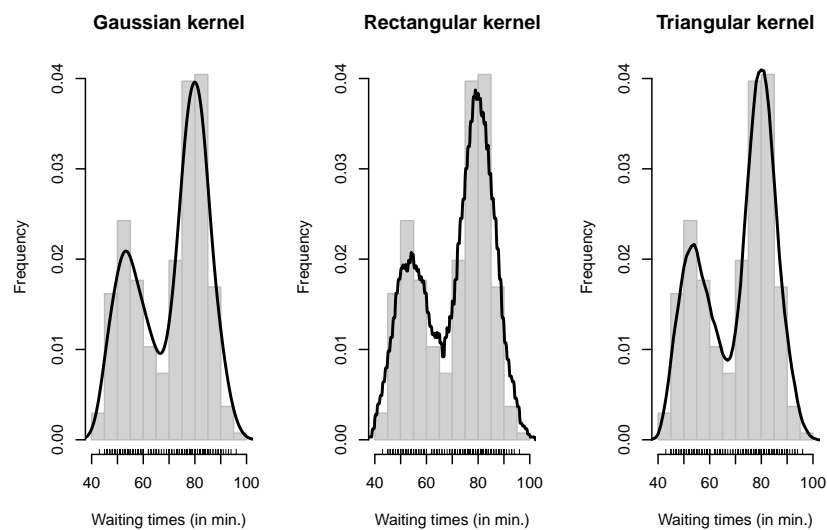
$value
[1] 1034

$counts
function gradient
      55         55

$convergence
[1] 0

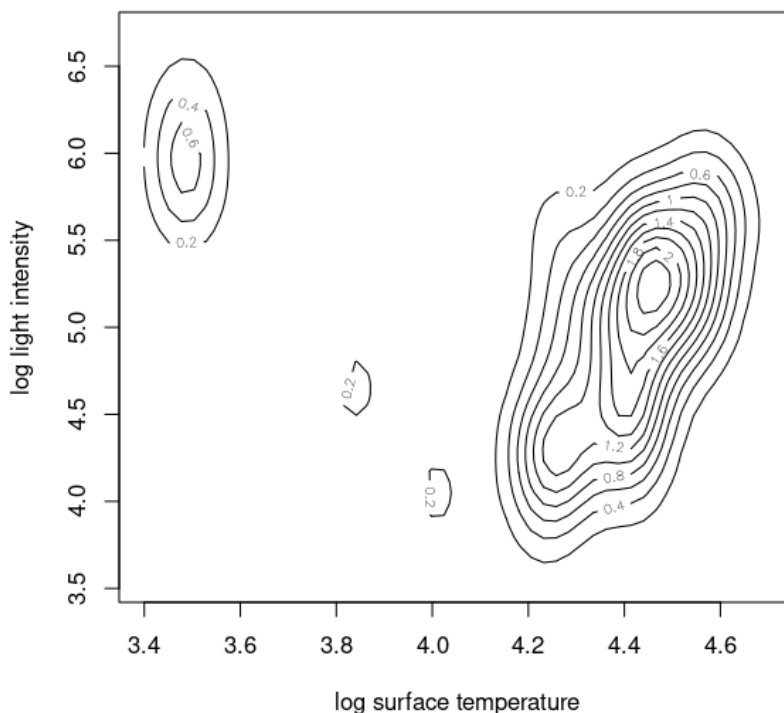
```

```
1 R> data("faithful", package = "datasets")
2 R> x <- faithful$waiting
3 R> layout(matrix(1:3, ncol = 3))
4 R> hist(x, xlab = "Waiting times (in min.)", ylab = "Frequency",
5 +       probability = TRUE, main = "Gaussian kernel",
6 +       border = "gray")
7 R> lines(density(x, width = 12), lwd = 2)
8 R> rug(x)
9 R> hist(x, xlab = "Waiting times (in min.)", ylab = "Frequency",
10 +      probability = TRUE, main = "Rectangular kernel",
11 +      border = "gray")
12 R> lines(density(x, width = 12, window = "rectangular"), lwd = 2)
13 R> rug(x)
14 R> hist(x, xlab = "Waiting times (in min.)", ylab = "Frequency",
15 +      probability = TRUE, main = "Triangular kernel",
16 +      border = "gray")
17 R> lines(density(x, width = 12, window = "triangular"), lwd = 2)
18 R> rug(x)
```



**Figure 8.4** Density estimates of the geyser eruption data imposed on a histogram of the data.

```
R> library("KernSmooth")
R> data("CYGOB1", package = "HSAUR3")
R> CYGOB1d <- bkde2D(CYGOB1, bandwidth = sapply(CYGOB1, dpik))
R> contour(x = CYGOB1d$x1, y = CYGOB1d$x2, z = CYGOB1d$fhat,
+         xlab = "log surface temperature",
+         ylab = "log light intensity")
```



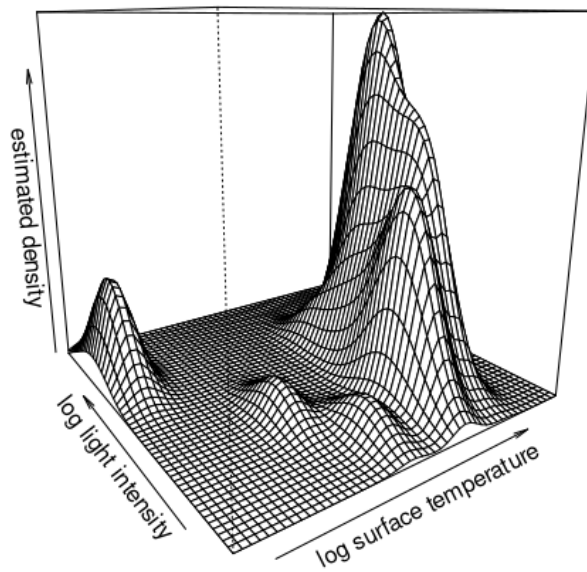
**Figure 8.5** A contour plot of the bivariate density estimate of the CYGOB1 data, i.e., a two-dimensional graphical display for a three-dimensional problem.

Of course, optimizing the appropriate likelihood ‘by hand’ is not very convenient. In fact, (at least) two packages offer high-level functionality for estimating mixture models. The first one is package **mclust** (Fraley et al., 2014) implementing the methodology described in Fraley and Raftery (2002). Here, a Bayesian information criterion (BIC) is applied to choose the form of the mixture model:

```
R> library("mclust")
```



```
R> persp(x = CYGOB1d$x1, y = CYGOB1d$x2, z = CYGOB1d$fhat,
+       xlab = "log surface temperature",
+       ylab = "log light intensity",
+       zlab = "estimated density",
+       theta = -35, axes = TRUE, box = TRUE)
```



**Figure 8.6** The bivariate density estimate of the CYGOB1 data, here shown in a three-dimensional fashion using the `persp` function.

```
R> mc <- Mclust(faithful$waiting)
R> mc

'Mclust' model object: (E,2)

Available components:
 [1] "call"          "data"          "modelName"
 [4] "n"             "d"             "G"
 [7] "BIC"           "loglik"        "df"
[10] "bic"           "ic1"           "hypvol"
[13] "parameters"   "z"             "classification"
[16] "uncertainty"
```

and the estimated means are

```
R> mc$parameters$mean
```

```
      1      2
54.6 80.1
```

with estimated standard deviation (found to be equal within both groups)

```
R> sqrt(mc$parameters$variance$sigma^2)
```

```
[1] 5.87
```

The proportion is  $\hat{p} = 0.36$ . The second package is called **flexmix** whose functionality is described by Leisch (2004). A mixture of two normals can be fitted using

```
R> library("flexmix")
```

```
R> fl <- flexmix(waiting ~ 1, data = faithful, k = 2)
```

with  $\hat{p} = 0.52$  and estimated parameters

```
R> parameters(fl, component = 1)
```

```
      Comp.1
coef.(Intercept) 70.8
sigma              13.6
```

```
R> parameters(fl, component = 2)
```

```
      Comp.2
coef.(Intercept) 71.0
sigma              13.6
```

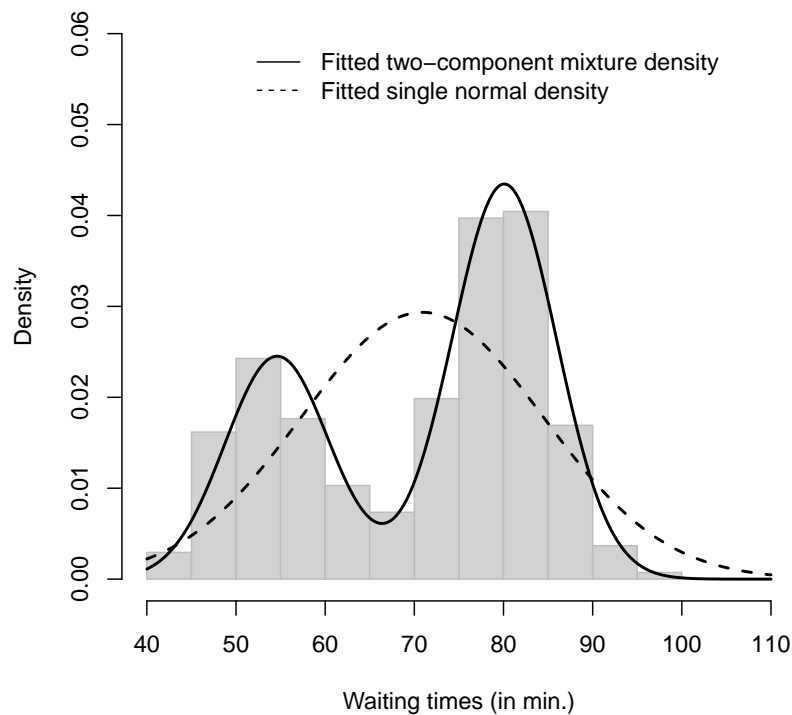
We can get standard errors for the five parameter estimates by using a bootstrap approach (see Efron and Tibshirani, 1993). The original data are slightly perturbed by drawing  $n$  out of  $n$  observations *with replacement* and those artificial replications of the original data are called *bootstrap samples*. Now, we can fit the mixture for each bootstrap sample and assess the variability of the estimates, for example using confidence intervals. Some suitable R code based on the `Mclust` function follows. First, we define a function that, for a bootstrap sample `indx`, fits a two-component mixture model and returns  $\hat{p}$  and the estimated means (note that we need to make sure that we always get an estimate of  $p$ , not  $1 - p$ ):

```
R> library("boot")
```

```
R> fit <- function(x, indx) {
+   a <- Mclust(x[indx], minG = 2, maxG = 2,
+             modelNames="E")$parameters
+   if (a$pro[1] < 0.5)
+     return(c(p = a$pro[1], mu1 = a$mean[1],
+             mu2 = a$mean[2]))
+   return(c(p = 1 - a$pro[1], mu1 = a$mean[2],
+             mu2 = a$mean[1]))
+ }
```

The function `fit` can now be fed into the `boot` function (Canty and Ripley, 2014) for bootstrapping (here 1000 bootstrap samples are drawn)

```
R> opar <- as.list(opp$par)
R> rx <- seq(from = 40, to = 110, by = 0.1)
R> d1 <- dnorm(rx, mean = opar$mu1, sd = opar$sd1)
R> d2 <- dnorm(rx, mean = opar$mu2, sd = opar$sd2)
R> f <- opar$p * d1 + (1 - opar$p) * d2
R> hist(x, probability = TRUE, xlab = "Waiting times (in min.)",
+      border = "gray", xlim = range(rx), ylim = c(0, 0.06),
+      main = "")
R> lines(rx, f, lwd = 2)
R> lines(rx, dnorm(rx, mean = mean(x), sd = sd(x)), lty = 2,
+      lwd = 2)
R> legend(50, 0.06, lty = 1:2, bty = "n",
+      legend = c("Fitted two-component mixture density",
+      "Fitted single normal density"))
```



**Figure 8.7** Fitted normal density and two-component normal mixture for geyser eruption data.

```
R> bootpara <- boot(faithful$waiting, fit, R = 1000)
```

We assess the variability of our estimates  $\hat{p}$  by means of adjusted bootstrap percentile (BCa) confidence intervals, which for  $\hat{p}$  can be obtained from

```
R> boot.ci(bootpara, type = "bca", index = 1)
```

```
BOOTSTRAP CONFIDENCE INTERVAL CALCULATIONS
Based on 1000 bootstrap replicates
```

```
CALL :
boot.ci(boot.out = bootpara, type = "bca", index = 1)
```

```
Intervals :
Level      BCa
95%      ( 0.304,  0.423 )
Calculations and Intervals on Original Scale
```

We see that there is a reasonable variability in the mixture model; however, the means in the two components are rather stable, as can be seen from

```
R> boot.ci(bootpara, type = "bca", index = 2)
```

```
BOOTSTRAP CONFIDENCE INTERVAL CALCULATIONS
Based on 1000 bootstrap replicates
```

```
CALL :
boot.ci(boot.out = bootpara, type = "bca", index = 2)
```

```
Intervals :
Level      BCa
95%      (53.4, 56.1 )
Calculations and Intervals on Original Scale
```

for  $\hat{\mu}_1$  and for  $\hat{\mu}_2$  from

```
R> boot.ci(bootpara, type = "bca", index = 3)
```

```
BOOTSTRAP CONFIDENCE INTERVAL CALCULATIONS
Based on 1000 bootstrap replicates
```

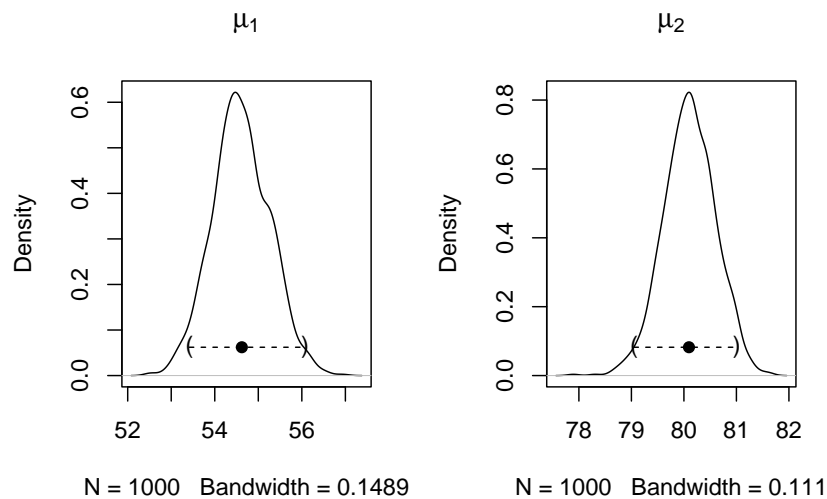
```
CALL :
boot.ci(boot.out = bootpara, type = "bca", index = 3)
```

```
Intervals :
Level      BCa
95%      (79, 81 )
Calculations and Intervals on Original Scale
```

Finally, we show a graphical representation of both the bootstrap distribution of the mean estimates *and* the corresponding confidence intervals. For convenience, we define a function for plotting, namely

```
R> bootplot <- function(b, index, main = "") {
+   dens <- density(b$t[,index])
+   ci <- boot.ci(b, type = "bca", index = index)$bca[4:5]
+   est <- b$t0[index]
+   plot(dens, main = main)
+   y <- max(dens$y) / 10
+   segments(ci[1], y, ci[2], y, lty = 2)
+   points(ci[1], y, pch = "(")
+   points(ci[2], y, pch = ")")
+   points(est, y, pch = 19)
+ }
```

```
R> layout(matrix(1:2, ncol = 2))
R> bootplot(bootpara, 2, main = expression(mu[1]))
R> bootplot(bootpara, 3, main = expression(mu[2]))
```



**Figure 8.8** Bootstrap distribution and confidence intervals for the mean estimates of a two-component mixture for the geyser data.

The element `t` of an object created by `boot` contains the bootstrap replications of our estimates, i.e., the values computed by `fit` for each of the 1000 bootstrap samples of the geyser data. First, we plot a simple density estimate and then construct a line representing the confidence interval. We apply this function to the bootstrap distributions of our estimates  $\hat{\mu}_1$  and  $\hat{\mu}_2$  in Figure 8.8.



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## Bibliography

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- Canty, A. and Ripley, B. D. (2014), *boot: Bootstrap R (S-PLUS) Functions*, URL <http://CRAN.R-project.org/package=boot>, R package version 1.3-9.
- Efron, B. and Tibshirani, R. J. (1993), *An Introduction to the Bootstrap*, London, UK: Chapman & Hall/CRC.
- Fraley, C. and Raftery, A. E. (2002), “Model-based clustering, discriminant analysis, and density estimation,” *Journal of the American Statistical Association*, 97, 611–631.
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- Silverman, B. (1986), *Density Estimation*, London, UK: Chapman & Hall/CRC.